Also solved by H. L. OLSON, ALBERT N. NAUER, JOSEPH B. REYNOLDS, C. E. FLANAGAN, G. PAASWELL, and HERBERT N. CARLETON.

2684 [March, 1918]. Proposed by B. J. BROWN, Kansas City, Missouri.

Find the locus of the center of a conic passing through four fixed points.

I. SOLUTION BY H. D. THOMPSON, Princeton University.

This is an exercise given in books on coördinate geometry.

Take the x-axis through two of the four points, and the y-axis, oblique, through the other two. Let $1/l$ and $1/l'$ be the abscissas of the two points on the x-axis and $1/m$ and $1/m'$, the ordinates of the two points on the y-axis. Then $lx + my - 1 = 0$ and $l'x + m'y - 1 = 0$ form another pair of lines, containing the four points in pairs. All conics through the four points are given by $\lambda xy + (lx + my - 1)(l'x + m'y - 1) = 0$, when λ is the parameter of the system.

The center of a representative conic is given by $2ll'x + (lm' + l'm)y - (l + l') + \lambda y = 0$ and $(lm' + l'm)x + 2mm'y - (m + m') + \lambda x = 0$. Eliminating λ, the locus of the center is $2ll'x^2 - 2mm'y^2 - (l + l')x + (m + m')y = 0$, a conic through the origin. The center of the locus is $\{1/4(1/l' + 1/l), 1/4(1/m' + 1/m)\}$. As any pair of opposite sides of the complete quadrilateral with the original four points as vertices can be taken as the axes, the locus of the centers will pass through the three points of intersection of opposite sides of the complete quadrilateral.

The locus is an hyperbola when $ll'mm'$ is positive, that is, when the original four points may be taken as the vertices of a convex polygon; and it is in this case only that the original conic may be a parabola (two).

II. SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The equation of a conic passing through the four given points $\pm\alpha_1$, $\pm\beta_1$, $\pm\gamma_1$, trilinear coördinates being used, is of the form

$$l_1\alpha^2 + m_1\beta^2 + n_1\gamma^2 = 0 \tag{1}$$

with the condition

$$l_1\alpha_1^2 + m_1\beta_1^2 + n_1\gamma_1^2 = 0. \tag{2}$$

The coördinates of the center of (1) are given by

$$\frac{l_1\alpha}{a} = \frac{m_1\beta}{b} = \frac{n_1\gamma}{c}, \tag{3}$$

a, b, c, being the sides of the fundamental triangle.

Eliminating l_1, m_1, n_1 from (3) and (2), we have

$$\frac{a\alpha_1^2}{\alpha} + \frac{b\beta_1^2}{\beta} + \frac{c\gamma_1^2}{c} = 0, \tag{4}$$

the required locus.

This is the nine-point conic of the quadrilateral whose vertices are the four given points.

Also solved by PAUL CAPRON and ELIJAH SWIFT.

2685 [March, 1918]. Proposed by JOSEPH B. REYNOLDS, Lehigh University.

A particle is describing an ellipse of eccentricity $\sqrt{2/3}$ as a central orbit about a focus when the attracting force suddenly becomes repulsive without changing its magnitude and the particle begins to describe an equilateral hyperbola; find where the change occurred and the angle that the major axis of the new orbit makes with that of the old orbit.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

The focal equation of the ellipse is

$$p_1^2 = \frac{a_1^2(1 - e_1^2)r_1}{2a_1 - r_1} \tag{1}$$

and of the hyperbola,

$$p_2^2 = \frac{a_2^2(e_2^2 - 1)r_2}{2a_2 + r_2} \tag{2}$$

an r being a radius vector; e, eccentricity; a, semi-major axis; p, perpendicular from a focus upon the corresponding tangent. In the ellipse, $e_1^2 = 2/3$, and in the hyperbola, $e_2^2 = 2$.

For the central force in (1),

$$F_1 = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{a_1(1-e_1^2)} \cdot \frac{1}{r_1^2}, \tag{3}$$

h being the usual constant in the theory of central forces.

By condition, we may write

$$-\frac{h^2}{a_1(1-e_1^2)} \cdot \frac{1}{r_2^2} = \frac{h^2}{p_2^3}\frac{dp_2}{dr_2}. \tag{4}$$

Integrating,

$$\frac{1}{a_1(1-e_1^2)} \cdot \frac{1}{r_2} = -\frac{1}{2p_2^2} + C \tag{5}$$

Let $r_2 = c = r_1$ at the instant of change in the nature of the force, when also

$$p_2^2 = p_1^2 = \frac{1}{a_1^2(1-e_1^2)}\frac{2a_1 - c}{c} \tag{6}$$

by (1); then

$$C = \frac{4a_1 - c}{2a_1^2(1-e_1^2)c} \tag{7}$$

and (5) becomes

$$\frac{1}{p_2^2} = \frac{(4a_1 - c)r_2 - 2a_1c}{a_1^2c(1-e_1^2)r_2} \tag{8}$$

or,

$$p_2^2 = \frac{a_1^2c(1-e_1^2)}{4a_1 - c} r_2 \div \left\{ r_2 - \frac{2a_1c}{4a_1 - c} \right\}$$

which is plainly the hyperbola in (2).

Now, (9) and (2) are identical if

$$a_2^2(e_2^2 - 1) = \frac{a_1^2c(1-e_1^2)}{4a_1 - c} \tag{10}$$

and

$$2a_2 = -\frac{2a_1c}{4a_1 - c}. \tag{11}$$

Eliminating a_2, and solving for c, $c = a_1$ (12), showing that the nature of the force changes at an extremity of the minor axis of the ellipse.

Let α = the angular coördinate of this point, the position of the center of force being the pole, γ = the required angle of the problem, and l_1, l_2, the latera recta of the curves; then the focal polar equations of the curves are

$$\frac{l_1}{r} = 1 - e_1 \cos\theta \quad (13), \qquad \frac{l_2}{r} = 1 - e_2\cos(\theta - \gamma) \quad (14).$$

The equations of the tangent to these at the common point α are

$$\frac{l_1}{r} = \cos(\theta - \alpha) - e_1\cos\theta \quad (15), \qquad \frac{l_2}{r} = \cos(\theta - \alpha) - e_2\cos(\theta-\gamma) \quad (16),$$

or, in cartesian coördinates,

$$(\cos\alpha - e_1)x + y\sin\alpha = l_1 \tag{17}$$

$$(\cos\alpha - e_2\cos\gamma)x + (\sin\alpha - e_2\sin\gamma)y = l_2. \tag{18}$$

These are identical if

$$\frac{\cos\alpha - e_1}{l_1} = \frac{\cos\alpha - e_2\cos\gamma}{l_2} \quad (19), \qquad \frac{\sin\alpha}{l_1} = \frac{\sin\alpha - e_2\sin\gamma}{l_2} \quad (20).$$

But from the ellipse, $\cos\alpha = e_1$, and (19) gives

$$\cos\gamma = \frac{e_1}{e_2} = \tfrac{1}{3}\sqrt{3}. \tag{2}$$

It may be added that the condition that (13) and (14) have the kind of contact consistent with the nature of the problem is

$$l_1^2(1 - e_2^2) + l_2^2(1 - e_1^2) - 2l_1l_2 + 2l_1l_2e_1e_2 \cos \gamma = 0. \tag{22}$$

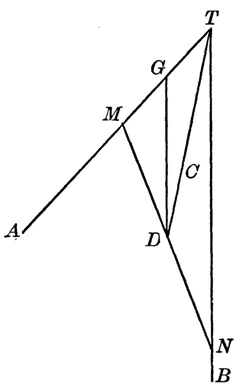
Also solved by HORACE L. OLSON.

2687 [March, 1918]. Proposed by N. P. PANDYA, Sojitra, India.

An ellipse intersects a parabola in A and B, and the tangents at A and B to the parabola meet at T. The center C of the ellipse lies within the space enclosed by the parabola and the tangents. Draw a third tangent to the parabola such that C may be the centroid of the triangle formed by the three tangents.

SOLUTION BY HORACE L. OLSON, Heidelberg University, Tiffin, Ohio.

This problem can be solved, if at all, without reference either to the ellipse or to the parabola. For this purpose let us alter the problem to read as follows: "Given two straight lines, TA and TB, intersecting at T, and a point C; draw a third line which shall form, with TA and TB, a triangle whose centroid shall be the point C." The problem, as thus stated, has a unique solution. If the third straight line is tangent to the parabola mentioned above, it is the solution of the original problem; if not, there is no solution. Draw the line TC, and extend it beyond C to D, so that $CD = \frac{1}{2}TC$. D will then be the mid-point of the third side of the required triangle. Through D, draw the line GD parallel to TB.



Lay off $GM = TG$. The line MDN, intersecting TB at N, will then be the required line; for, since the line GD is parallel to the side TN of the triangle TMN and bisects the side TM, it must also bisect the side MN. Since this demonstration is reversible, MN is the only line-segment included between the lines TA and TB, and bisected at D. Hence, if MN is tangent to the given parabola, it is the solution of the original problem; if not, there is no solution.

Also solved analytically by WILLIAM HOOVER.

2691 [April, 1918]. Proposed by ROGER A. JOHNSON, Hamline University.

Show by purely geometric methods, without the use of the calculus, that the envelope of the circles whose centers are on a fixed circle and which touch a fixed diameter of that circle is a two-arched epicycloid. Cf. problem 423 (calculus) [February and September, 1917].

SOLUTION BY THE PROPOSER.

First, let us make a few general remarks about the envelope of a system of circles. In general, the points where one of a system of curves meets the envelope of the system are the limiting positions of the points of intersection of that curve with a near-by one, as the latter comes into coincidence with the former. But two circles generally intersect in two points; the perpendicular bisector of the line connecting them is the line joining the centers of the circles. Hence,